

ALGEBRAIC-GEOMETRIC CHARACTERIZATION OF AUTONOMY AND CONTROLLABILITY OF BEHAVIOURS OF SPATIALLY INVARIANT SYSTEMS

AMOL SASANE

ABSTRACT. We give algebraic-geometric characterizations of the properties of autonomy and of controllability of behaviours of spatially invariant dynamical systems, consisting of distributional solutions w , that are periodic in the spatial variables, to a partial differential equation

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right)w = 0,$$

corresponding to a polynomial $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$.

1. INTRODUCTION

Consider a homogeneous, linear, constant coefficient partial differential equation, in \mathbb{R}^{d+1} described by a polynomial $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$:

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right)w = 0. \quad (1.1)$$

(That is, the operator

$$p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right)$$

is obtained from the polynomial $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ by making the replacements

$$\xi_k \rightsquigarrow \frac{\partial}{\partial x_k} \text{ for } k = 1, \dots, d, \text{ and } \tau \rightsquigarrow \frac{\partial}{\partial t}.$$

In the behavioural approach to control theory, the “behaviour” $\mathfrak{B}_{\mathcal{W}}(p)$ associated with p in \mathcal{W} (where \mathcal{W} is an appropriate solution space, for example smooth functions $C^\infty(\mathbb{R}^{d+1})$ or distribution spaces like $\mathcal{D}'(\mathbb{R}^{d+1})$ or $\mathcal{S}'(\mathbb{R}^{d+1})$ and so on), is defined to be the set of all solutions $w \in \mathcal{W}$ that satisfy the above PDE (1.1), and one characterizes algebraically (in terms of algebraic properties of the polynomial p) certain analytical properties of $\mathfrak{B}_{\mathcal{W}}(p)$ (for example, the control theoretic properties of autonomy, controllability, stability, and so on). See for example [8], [9], [1] for distinct takes on this in the context of systems described by partial differential equations.

For example let us consider the property of “autonomy”, which means the following.

2010 *Mathematics Subject Classification.* Primary 35A24; Secondary 93B05, 93C20.

Key words and phrases. partial differential equations, distributions that are periodic in the spatial directions, Fourier transformation, behaviours, autonomy, controllability, spatially invariant systems.

Definition 1.1. Let \mathcal{W} be a subspace of $\mathcal{D}'(\mathbb{R}^{d+1})$ which is invariant under differentiation, that is, for all $w \in \mathcal{W}$,

$$\frac{\partial}{\partial x_k} w \in \mathcal{W} \text{ for } k = 1, \dots, d, \text{ and } \frac{\partial}{\partial t} w \in \mathcal{W}.$$

If $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$, then the *behaviour* (of p in \mathcal{W}) is

$$\mathfrak{B}_{\mathcal{W}}(p) := \left\{ w \in \mathcal{W} : p \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0 \right\}.$$

We call the behaviour $\mathfrak{B}_{\mathcal{W}}(p)$ *autonomous* (with respect to \mathcal{W}) if the only $w \in \mathfrak{B}_{\mathcal{W}}(p)$ satisfying $w|_{t < 0} = 0$ is $w = 0$.

The following [9, Theorem 3.4] is an easy consequence of [7, p.310, Theorem 8.6.7].

Proposition 1.2. Let $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ be a nonzero polynomial. Then the behaviour

$$\mathfrak{B}_{\mathcal{D}'(\mathbb{R}^{d+1})}(p) := \left\{ w \in \mathcal{D}'(\mathbb{R}^{d+1}) : p \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0 \right\}$$

corresponding to p is autonomous if and only if $\deg p = \deg p(\mathbf{0}, \tau)$.

Here by $\deg(\cdot)$, we mean the *total degree*, which is the maximum (over the monomials occuring in the polynomial) of the sum of the degrees of the exponents of indeterminates in the monomial. Also, $p(\mathbf{0}, \tau)$ denotes the polynomial in $\mathbb{C}[\tau]$ obtained from $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ by making the substitutions $\xi_k \mapsto 0$ for $k = 1, \dots, d$.

There has been recent interest in “spatially invariant systems”, see for example [3], [4], where one considers solutions to PDEs that are periodic along the spatial direction. So it is a natural question to ask what the analogue of Proposition 1.2 is, when we replace the solution space $\mathcal{D}'(\mathbb{R}^{d+1})$ with one that consists only of those solutions that are *periodic* in the spatial directions. In this article, our first main result is the following one, characterizing autonomy of spatially invariant systems.

Theorem 1.3. Let $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ be a nonzero polynomial and $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ be a linearly independent set vectors in \mathbb{R}^d . Then the behaviour

$$\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p) := \left\{ w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) : p \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0 \right\}$$

is autonomous if and only if $(\mathcal{V}(C_{\xi}(p))) \cap (2\pi i A^{-1} \mathbb{Z}^d) = \emptyset$, where A is the matrix with its rows equal to the transposes of the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_d$:

$$A := \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_d^\top \end{bmatrix}.$$

Here $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ is, roughly speaking, the set of all distributions on \mathbb{R}^{d+1} that are periodic in the spatial direction with a discrete set \mathbb{A} of periods. The precise definition of $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ is given in Subsection 1.1. Also, in the above result, the condition

$$(\mathcal{V}(C_{\xi}(p))) \cap (2\pi i A^{-1} \mathbb{Z}^d) = \emptyset$$

is an algebraic-geometric condition, saying that the variety $\mathcal{V}(C_{\xi}(p))$ of the “ ξ -content” of p does not meet the discrete set of points in $2\pi i A^{-1} \mathbb{Z}^d$. The notion of the ξ -content of a polynomial is defined in Subsection 1.2.

1.1. **The space $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$.** For $\mathbf{a} \in \mathbb{R}^d$, the *translation operation* $\mathbf{S}_{\mathbf{a}}$ on distributions in $\mathcal{D}'(\mathbb{R}^d)$ is defined by

$$\langle \mathbf{S}_{\mathbf{a}}(T), \varphi \rangle = \langle T, \varphi(\cdot + \mathbf{a}) \rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}^d).$$

A distribution $T \in \mathcal{D}'(\mathbb{R}^d)$ is said to be *periodic with a period $\mathbf{a} \in \mathbb{R}^d$* if $T = \mathbf{S}_{\mathbf{a}}(T)$.

Let $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ be a linearly independent set vectors in \mathbb{R}^d . We define $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ to be the set of all distributions T that satisfy

$$\mathbf{S}_{\mathbf{a}_k}(T) = T, \quad k = 1, \dots, d.$$

From [5, §34], T is a tempered distribution, and from the above it follows by taking Fourier transforms that $(1 - e^{2\pi i \mathbf{a}_k \cdot \mathbf{y}}) \widehat{T} = 0$ for $k = 1, \dots, d$. It can be seen that

$$\widehat{T} = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \alpha_{\mathbf{v}}(T) \delta_{\mathbf{v}},$$

for some scalars $\alpha_{\mathbf{v}} \in \mathbb{C}$, and where A is the matrix with its rows equal to the transposes of the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_d$:

$$A := \begin{bmatrix} \mathbf{a}_1^\top \\ \vdots \\ \mathbf{a}_d^\top \end{bmatrix}.$$

Also, in the above, $\delta_{\mathbf{v}}$ denotes the usual Dirac measure with support in \mathbf{v} :

$$\langle \delta_{\mathbf{v}}, \psi \rangle = \psi(\mathbf{v}) \quad \text{for } \psi \in \mathcal{D}'(\mathbb{R}^d).$$

By the Schwartz Kernel Theorem (see for instance [7, p. 128, Theorem 5.2.1]), $\mathcal{D}'(\mathbb{R}^{d+1})$ is isomorphic as a topological space to $\mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$, the space of all continuous linear maps from $\mathcal{D}(\mathbb{R})$ to $\mathcal{D}'(\mathbb{R}^d)$, thought of as vector-valued distributions. For preliminaries on vector-valued distributions, we refer the reader to [2]. We indicate this isomorphism by putting an arrow on top of elements of $\mathcal{D}'(\mathbb{R}^{d+1})$. Thus for $w \in \mathcal{D}'(\mathbb{R}^{d+1})$, we set $\vec{w} \in \mathcal{L}(\mathcal{D}(\mathbb{R}), \mathcal{D}'(\mathbb{R}^d))$ to be the vector valued distribution defined by

$$\langle \vec{w}(\varphi), \psi \rangle = \langle w, \psi \otimes \varphi \rangle$$

for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. We define

$$\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) = \{w \in \mathcal{D}'(\mathbb{R}^{d+1}) : \text{for all } \varphi \in \mathcal{D}(\mathbb{R}), \vec{w}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)\}.$$

Then for $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$,

$$\frac{\partial}{\partial x_k} w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) \quad \text{for } k = 1, \dots, d, \quad \text{and} \quad \frac{\partial}{\partial t} w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}).$$

Also, for $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, we define $\widehat{w} \in \mathcal{D}'(\mathbb{R}^{d+1})$ by

$$\langle \widehat{w}, \psi \otimes \varphi \rangle = \langle \vec{w}(\varphi), \widehat{\psi} \rangle,$$

for $\varphi \in \mathcal{D}(\mathbb{R})$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. That this specifies a well-defined distribution in $\mathcal{D}'(\mathbb{R}^{d+1})$, can be seen using the fact that for every $\Phi \in \mathcal{D}(\mathbb{R}^{d+1})$, there exists a sequence of functions $(\Psi_n)_n$ that are finite sums of direct products of test functions, that is, $\Psi_n = \sum_k \psi_k \otimes \varphi_k$, where $\psi_k \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi_k \in \mathcal{D}(\mathbb{R})$, such that Ψ_n converges to Φ in $\mathcal{D}(\mathbb{R}^{d+1})$. We also have

$$\widehat{\frac{\partial}{\partial x_k} w} = 2\pi i y_k \widehat{w} \quad \text{for } k = 1, \dots, d, \quad \text{and} \quad \widehat{\frac{\partial}{\partial t} w} = \frac{\partial}{\partial t} \widehat{w}.$$

Here $\mathbf{y} = (y_1, \dots, y_d)$ is the Fourier transform variable.

1.2. The ξ -content of a polynomial and its variety. If p is a polynomial in $\mathbb{C}[\xi_1, \dots, \xi_d, \tau]$, then we can write

$$p = c_0 + c_1\tau + c_2\tau^2 + \dots + c_n\tau^n \in \mathbb{C}[\xi_1, \dots, \xi_d][\tau],$$

where $c_0, c_1, c_2, \dots, c_n \in \mathbb{C}[\xi_1, \dots, \xi_d]$. The ξ -content $C_\xi(p)$ is defined as the ideal in the polynomial ring $\mathbb{C}[\xi_1, \dots, \xi_d]$ generated by $c_0, c_1, c_2, \dots, c_n$.

Given a set I of polynomials from $\mathbb{C}[\xi_1, \dots, \xi_d]$, the notation $\mathcal{V}(I)$ means the (*affine*) *variety* (in \mathbb{C}^d) of I , that is, the set of all common zeros of each polynomial in I :

$$\mathcal{V}(I) = \{\mathbf{v} \in \mathbb{C}^d : p(\mathbf{v}) = 0 \text{ for all } p \in I\}.$$

Thus given a $p = c_0 + c_1\tau + c_2\tau^2 + \dots + c_n\tau^n \in \mathbb{C}[\xi_1, \dots, \xi_d][\tau]$, we have

$$\mathcal{V}(C_\xi(p)) = \{\mathbf{v} \in \mathbb{C}^d : c_0(\mathbf{v}) = \dots = c_n(\mathbf{v}) = 0\}.$$

1.3. Controllability.

Definition 1.4. Let \mathcal{W} be a subspace of $\mathcal{D}'(\mathbb{R}^{d+1})$ which is invariant under differentiation and $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$. Then we call $\mathfrak{B}_{\mathcal{W}}(p)$ *controllable* if for every $w_1, w_2 \in \mathfrak{B}_{\mathcal{W}}(p)$, there is a $w \in \mathfrak{B}_{\mathcal{W}}(p)$ and a $\tau \geq 0$ such that $w|_{(-\infty, 0)} = w_1|_{(-\infty, 0)}$ and $w|_{(\tau, +\infty)} = w_2|_{(\tau, +\infty)}$.

The second main result in this article is the following one, characterizing controllability of spatially invariant systems:

Theorem 1.5. Let $p \in \mathbb{C}[\xi_1, \dots, \xi_d, \tau]$ be a nonzero polynomial and $\mathbb{A} := \{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ be a linearly independent set vectors in \mathbb{R}^d . Let

$$\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p) := \left\{ w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1}) : p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right) w = 0 \right\} \neq \{0\}.$$

Then the following are equivalent:

- (1) $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is controllable.
- (2) $c_0(2\pi i \mathbf{v}) = \dots = c_n(2\pi i \mathbf{v}) = 0$ for all $\mathbf{v} \in A^{-1}\mathbb{Z}^d$, where for $k = 0, 1, \dots, n$, the polynomial $c_k \in \mathbb{C}[\xi_1, \dots, \xi_d]$ is the coefficient of τ^k in the decomposition of p as $p = c_0 + c_1\tau + \dots + c_n\tau^n$, and A is the matrix with its rows equal to the transposes of the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_d$.

2. PROOF OF THEOREM 1.3

Before we prove our main result, we illustrate the basic idea behind ‘If’ part: by taking Fourier transform, the partial derivatives with respect to the spatial variables are converted into the polynomial coefficients $a_k(2\pi i \mathbf{y})$, where \mathbf{y} is the vector of Fourier transform variables y, \dots, y_d . But the support of \hat{w} is carried on a family of lines, indexed by $\mathbf{n} \in \mathbb{Z}^d$, in \mathbb{R}^{d+1} parallel to the time axis. So we obtain a family of ODEs, parameterized by $\mathbf{n} \in \mathbb{Z}^d$, and by “freezing” an $\mathbf{n} \in \mathbb{Z}^d$, we get an ODE, where for a solution we can indeed say that zero past implies a zero future, and so the proof can be completed easily.

Proof of Theorem 1.3. ‘Only if’ part: Let $\mathbf{v} \in \mathbb{C}^d$ be such that

$$\mathbf{v} \in \mathcal{V}(C_\xi(p)) \cap \left(2\pi i A^{-1}\mathbb{Z}^d\right).$$

Consider $w := e^{\mathbf{v} \cdot \mathbf{x}} \otimes \Theta$, where Θ is any nonzero function in $C^\infty(\mathbb{R})$ which has a zero past. Here the notation \cdot is used for the usual Euclidean inner product in the complex vector space \mathbb{C}^d :

thus we have $\mathbf{v} \cdot \mathbf{x} := v_1 x_1 + \dots + v_d x_d$, where $\mathbf{v} = (v_1, \dots, v_d) \in \mathbb{C}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Take for example Θ given by

$$\Theta(t) = \begin{cases} 0 & \text{if } t \leq 0, \\ e^{-1/t} & \text{if } t > 0. \end{cases}$$

Then we have that $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$, since

$$\mathbf{S}_{\mathbf{a}_k} w = e^{\mathbf{v} \cdot (\mathbf{x} + \mathbf{a}_k)} \otimes \Theta = e^{\mathbf{v} \cdot \mathbf{a}_k} e^{\mathbf{v} \cdot \mathbf{x}} \otimes \Theta = 1 \cdot e^{\mathbf{v} \cdot \mathbf{x}} \otimes \Theta = w.$$

Moreover, w has zero past, that is, $w|_{t < 0} = 0$ because $\Theta|_{t < 0} = 0$. If $p = c_0 + c_1 \tau + \dots + c_n \tau^n$, where $c_0, c_1, \dots, c_n \in \mathbb{C}[\xi_1, \dots, \xi_d]$, then $c_0, \dots, c_n \in C_{\xi}(p)$, and so $c_0(\mathbf{v}) = \dots = c_n(\mathbf{v}) = 0$. Using this, we can see that $w \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ because

$$p \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = \left(c_0(\mathbf{v})\Theta + c_1(\mathbf{v})\Theta' + \dots + c_n(\mathbf{v})\Theta^{(n)} \right) e^{\mathbf{v} \cdot \mathbf{x}} = 0.$$

Consequently, $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is not autonomous. This completes the proof of the ‘only if’ part.

‘If’ part: Suppose that $\mathcal{V}(C_{\xi}(p)) \cap (2\pi i A^{-1} \mathbb{Z}^d) = \emptyset$. Let $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ satisfy $w|_{t < 0} = 0$ and

$$p \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t} \right) w = 0. \quad (2.1)$$

Let $p = c_0 + c_1 \tau + \dots + c_n \tau^n \in \mathbb{C}[\xi_1, \dots, \xi_d][\tau]$, where $c_0, c_1, c_2, \dots, c_n \in \mathbb{C}[\xi_1, \dots, \xi_d]$ and $c_n \neq 0$ as a polynomial in $\mathbb{C}[\xi_1, \dots, \xi_d]$. Upon taking Fourier transformation on both sides of the equation (2.1) with respect to the spatial variables, we obtain

$$c_0(2\pi i \mathbf{y}) \hat{w} + c_1(2\pi i \mathbf{y}) \frac{\partial}{\partial t} \hat{w} + \dots + c_n(2\pi i \mathbf{y}) \left(\frac{\partial}{\partial t} \right)^n \hat{w} = 0. \quad (2.2)$$

For each fixed $\varphi \in \mathcal{D}(\mathbb{R})$, $\vec{w} \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$, and so it follows that

$$\hat{w} = \sum_{\mathbf{v} \in A^{-1} \mathbb{Z}^d} \alpha_{\mathbf{v}}(\hat{w}, \varphi) \delta_{\mathbf{v}}, \quad (2.3)$$

for appropriate scalars $\alpha_{\mathbf{v}}(T, \varphi) \in \mathbb{C}$. In particular, we see that the support of \hat{w} is contained in $A^{-1} \mathbb{Z}^d \times [0, +\infty)$. Thus each of the half lines in $A^{-1} \mathbb{Z}^d \times [0, +\infty)$ carries a solution of the differential equation (2.2), and \hat{w} is a sum of these. We will show that each of these summands is zero. It follows from (2.3) that the map $\varphi \mapsto \alpha_{\mathbf{v}}(\hat{w}, \varphi) : \mathcal{D}(\mathbb{R}) \rightarrow \mathbb{C}$ defines a distribution $T_{\mathbf{v}}$ in $\mathbb{D}'(\mathbb{R})$. Moreover, the support of $T_{\mathbf{v}}$ is contained in $[0, +\infty)$. From (2.2), we see that in a small enough neighbourhood of $\mathbf{v} \in A^{-1} \mathbb{Z}^d$, we have

$$c_0(2\pi i \mathbf{v}) T_{\mathbf{v}} + c_1(2\pi i \mathbf{v}) \frac{d}{dt} T_{\mathbf{v}} + \dots + c_n(2\pi i \mathbf{v}) \left(\frac{d}{dt} \right)^n T_{\mathbf{v}} = 0.$$

Owing to our hypothesis that $\mathcal{V}(C_{\xi}(p)) \cap (2\pi i A^{-1} \mathbb{Z}^d) = \emptyset$, we know that at least one of the coefficients $c_0(2\pi i \mathbf{v}), \dots, c_n(2\pi i \mathbf{v})$ is nonzero. Together with the fact that $T_{\mathbf{v}}|_{(-\infty, 0)} = 0$, this yields that $T_{\mathbf{v}} = 0$ on \mathbb{R} . Hence $\hat{w} = 0$ in a neighbourhood of $\mathbf{v} \in A^{-1} \mathbb{Z}^d$. As the choice of $\mathbf{v} \in A^{-1} \mathbb{Z}^d$ was arbitrary, (2.3) implies that $\hat{w} = 0$ and so $w = 0$. Consequently, the behaviour $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is autonomous. \square

3. PROOF OF THEOREM 1.5

Proof of Theorem 1.5. (1) \Rightarrow (2): Suppose that $\mathfrak{B}_{\mathcal{W}}(p)$ is controllable. Let $\mathbf{v} \in A^{-1}\mathbb{Z}^d$ be such that $c_k(2\pi i\mathbf{v}) \neq 0$ for some $k \in \{0, 1, \dots, n\}$, and $c_\ell(2\pi i\mathbf{v}) = 0$ for all $\ell > k$. Let $\tau > 0$, and let $\Theta \in C^\infty(\mathbb{R})$ be any solution to the ODE

$$c_0(2\pi i\mathbf{v})\Theta + c_1(2\pi i\mathbf{v})\Theta' + \dots + c_k(2\pi i\mathbf{v})\Theta^{(k)} = 0.$$

and $\Theta|_{(\tau, +\infty)} \neq 0$. Define $w_2 \in \mathcal{D}'(\mathbb{R}^{d+1})$ by $w_2 := e^{2\pi i\mathbf{v} \cdot x} \otimes \Theta$. Then $w_2 \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$ since for all $k \in \{1, \dots, d\}$, we have $\mathbf{S}_{\mathbf{a}_k} w_2 = e^{2\pi i\mathbf{v} \cdot (\mathbf{x} + \mathbf{a}_k)} \otimes \Theta = e^{2\pi i\mathbf{v} \cdot \mathbf{a}_k} e^{2\pi i\mathbf{v} \cdot \mathbf{x}} \otimes \Theta = 1 \cdot e^{2\pi i\mathbf{v} \cdot \mathbf{x}} \otimes \Theta = w_2$. Also, $w_2 \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ because

$$\begin{aligned} p\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d}, \frac{\partial}{\partial t}\right) w_2 &= \left(c_0(2\pi i\mathbf{v})\Theta + c_1(2\pi i\mathbf{v})\Theta' + \dots + c_k(2\pi i\mathbf{v})\Theta^{(k)} + 0\right) e^{2\pi i\mathbf{v} \cdot \mathbf{x}} \\ &= 0 \cdot e^{2\pi i\mathbf{v} \cdot \mathbf{x}} = 0. \end{aligned}$$

Suppose w patches up $w_1 := 0$ and w_2 , that is, $w \in \mathfrak{B}_{\mathcal{W}}(p)$ satisfies $w|_{(-\infty, 0)} = w_1|_{(-\infty, 0)} = 0$ and $w|_{(\tau, +\infty)} = w_2|_{(\tau, +\infty)} \neq 0$. Proceeding as in the proof of the ‘if’ part of Theorem 1.3,

$$\widehat{w} = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T^{(\mathbf{v})},$$

for some $T^{(\mathbf{v})} \in \mathcal{D}'(\mathbb{R})$. Thus we have

$$c_0(2\pi i\mathbf{v})T + c_1(2\pi i\mathbf{v})\frac{d}{dt}T + \dots + c_k(2\pi i\mathbf{v})\left(\frac{d}{dt}\right)^k T = 0,$$

and $T|_{(-\infty, 0)} = 0$, while $T|_{(\tau, +\infty)} = \Theta|_{(\tau, +\infty)} \neq 0$, which is impossible, since $c_k(2\pi i\mathbf{v}) \neq 0$. This completes the proof of the fact that (1) \Rightarrow (2).

(2) \Rightarrow (1): Let us suppose that $w_1, w_2 \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$. Then we have

$$\widehat{w}_1 = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T_1^{(\mathbf{v})}, \quad \widehat{w}_2 = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T_2^{(\mathbf{v})},$$

for some $T_1^{(\mathbf{v})}, T_2^{(\mathbf{v})} \in \mathcal{D}'(\mathbb{R})$. Moreover, owing to the correspondence between $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$ and the space of sequences $s'(\mathbb{Z}^d)$ of at most polynomial growth, it follows that for each $\varphi \in \mathcal{D}(\mathbb{R})$, there exist $M_\varphi > 0$ and a positive integer k_φ , such that we have the estimates

$$|\langle T_1^{(\mathbf{v})}, \varphi \rangle| \leq M_\varphi(1 + |\mathbf{n}|)^{k_\varphi}, \quad |\langle T_2^{(\mathbf{v})}, \varphi \rangle| \leq M_\varphi(1 + |\mathbf{n}|)^{k_\varphi},$$

for all $\mathbf{n} := A\mathbf{v} \in \mathbb{Z}^d$. Let $\tau > 0$, and let $\theta \in C^\infty(\mathbb{R})$ be such that $\theta(t) = 1$ for all $t \leq 0$, $\theta(t) = 0$ for all $t > \tau/4$ and $0 \leq \theta(t) \leq 1$ for all $t \in \mathbb{R}$. Define $T^{(\mathbf{v})} \in \mathcal{D}'(\mathbb{R})$ by

$$T^{(\mathbf{v})} := \theta T_1^{(\mathbf{v})} + \theta(\tau - \cdot) T_2^{(\mathbf{v})}.$$

Set $\widehat{w} \in \mathcal{D}'(\mathbb{R}^{d+1})$ to be $\widehat{w} = \sum_{\mathbf{v} \in A^{-1}\mathbb{Z}^d} \delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}$. Then for every $\varphi \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned} |\langle T^{(\mathbf{v})}, \varphi \rangle| &\leq |\langle \theta T_1^{(\mathbf{v})}, \varphi \rangle| + |\langle \theta(\tau - \cdot) T_2^{(\mathbf{v})}, \varphi \rangle| \\ &\leq M_{\theta\varphi}(1 + |\mathbf{n}|)^{k_{\theta\varphi}} + M_{\theta(\tau - \cdot)\varphi}(1 + |\mathbf{n}|)^{k_{\theta(\tau - \cdot)\varphi}} \\ &\leq \max\{M_{\theta\varphi}, M_{\theta(\tau - \cdot)\varphi}\}(1 + |\mathbf{n}|)^{\max\{k_{\theta\varphi}, k_{\theta(\tau - \cdot)\varphi}\}}, \end{aligned}$$

and so $\vec{w}(\varphi) \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$. Thus $w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})$. Also, $w \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ because

$$\begin{aligned} & c_0(2\pi i \mathbf{y})(\delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}) + c_1(2\pi i \mathbf{y}) \frac{\partial}{\partial t}(\delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}) + \cdots + c_n(2\pi i \mathbf{y}) \left(\frac{\partial}{\partial t} \right)^n (\delta_{\mathbf{v}} \otimes T^{(\mathbf{v})}) \\ &= c_0(2\pi i \mathbf{v}) \delta_{\mathbf{v}} \otimes T^{(\mathbf{v})} + c_1(2\pi i \mathbf{v}) \delta_{\mathbf{v}} \otimes \frac{d}{dt} T^{(\mathbf{v})} + \cdots + c_n(2\pi i \mathbf{v}) \delta_{\mathbf{v}} \otimes \left(\frac{d}{dt} \right)^n T^{(\mathbf{v})} \\ &= 0, \end{aligned}$$

and so

$$c_0(2\pi i \mathbf{y}) \hat{w} + c_1(2\pi i \mathbf{y}) \frac{\partial}{\partial t} \hat{w} + \cdots + c_n(2\pi i \mathbf{y}) \left(\frac{\partial}{\partial t} \right)^n \hat{w} = 0.$$

Finally, because $T^{(\mathbf{v})}|_{(-\infty,0)} = T_1^{(\mathbf{v})}|_{(-\infty,0)}$ and $T^{(\mathbf{v})}|_{(\tau,+\infty)} = T_2^{(\mathbf{v})}|_{(\tau,+\infty)}$, it follows that $\hat{w}|_{(-\infty,0)} = \hat{w}_1|_{(-\infty,0)}$ and $\hat{w}|_{(\tau,+\infty)} = \hat{w}_2|_{(\tau,+\infty)}$. Consequently, $w|_{(-\infty,0)} = w_1|_{(-\infty,0)}$ and $w|_{(\tau,+\infty)} = w_2|_{(\tau,+\infty)}$, showing that the behaviour is controllable. This completes the proof. \square

Example 3.1 (Diffusion equation). Consider the diffusion equation

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_d^2} \right) w = 0,$$

corresponding to the polynomial $p = \tau - \xi_1^2 - \cdots - \xi_d^2$. Here the ξ -content $C_{\xi}(p)$ is the ideal in $\mathbb{C}[\xi, \dots, \xi_d]$ generated by the two polynomials 1 and $-\xi_1^2 - \cdots - \xi_d^2$, and so $C_{\xi}(p)$ is the full ring $\mathbb{C}[\xi, \dots, \xi_d]$. Consequently, its variety in \mathbb{C}^d is empty. Hence the behaviour $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is always autonomous, no matter what \mathbb{A} is. Note that this is in striking contrast to what happens when we look at just distributional solutions: since

$$\begin{aligned} \deg(p(\xi_1, \dots, \xi_d, \tau)) &= \deg(\tau - (\xi_1^2 + \cdots + \xi_d^2)) = 2 \\ &\quad \Downarrow \\ \deg(p(0, \dots, 0, \tau)) &= \deg(\tau) = 1 \end{aligned}$$

we have from Proposition 1.2 that $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is *not* autonomous, and this outcome is physically unexpected. Indeed, if we imagine the case of diffusion of heat, in which case the w is the temperature, say along a metallic rod when $d = 1$, then zero temperature upto time $t = 0$ should mean that the temperature stays zero in the future as well (since the above PDE describes the situation when no external heat is supplied). However, when one considers distributional solutions, one can have pathological solutions with a zero past that are nonzero in the future! But if we choose the physically “correct” solution space in this context, namely functions which at each time instant have a spatial profile belonging to $L^\infty(\mathbb{R})$, then it can be shown that solutions that are zero in the past are also zero in the future, as expected. So the real reason for the nonautonomy when one considers solutions in $\mathcal{D}'(\mathbb{R}^{d+1})$ is that there is no restriction on the spatial profiles of the solutions at each time instant, and wild growth (such as something which grows faster than $e^{|\mathbf{x}|^2}$) is allowed. However, with a *periodic* profile in the spatial direction, namely when the spatial profile is in $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d)$, we know that the spatial profile is automatically tempered (see for example [5]), and as we have seen above, in this case the behaviour $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is autonomous, in conformity with our physical expectation.

Similarly, since $c_0 = 1 \neq 0$, it follows from Theorem 1.5 that $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ is never controllable. \diamond

Remark 3.2. We end with an open question: Is there an algebraic-geometric characterization in terms of p of *approximate* controllability of $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$? Here, by approximate controllability, we mean the following.

Let $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d \times (0, +\infty)) := \{w|_{t>0} : w \in \mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})\}$, endowed with the induced topology from $\mathcal{D}'(\mathbb{R}^d \times (0, +\infty))$. We call $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ *approximately controllable* if for every $w_1, w_2 \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$, and every neighbourhood N of 0 in $\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^d \times (0, +\infty))$, there is a $w \in \mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$ and a $\tau \geq 0$ such that

$$w|_{(-\infty, 0)} = w_1|_{(-\infty, 0)} \quad \text{and} \\ \mathbf{S}_{(\mathbf{0}, -\tau)}(w - w_2)|_{(0, +\infty)} \in N.$$

Here, $\mathbf{S}_{(\mathbf{0}, -\tau)}$ denotes the translation operator in $\mathcal{D}'(\mathbb{R}^{d+1})$ corresponding to $(\mathbf{0}, -\tau) \in \mathbb{R}^{d+1}$.

When one considers just *smooth* solutions, that is, $\mathcal{W} = C^\infty(\mathbb{R}^{d+1})$, then one can define approximate controllability analogously to the above. (In particular, then the N in the definition is a neighbourhood in the appropriate topology of the Frechet space of C^∞ functions in the half-space $t > 0$.) Using the main result in [6], a characterization of approximate controllability in the smooth solution case was given in [9, Theorem 4.6].

It is not hard to show that for a nontrivial behaviour $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$, one has the following heirarchy of properties:

$$\text{controllability} \Rightarrow \text{approximate controllability} \Rightarrow \neg(\text{autonomy}).$$

Thus in our search for the appropriate algebraic-geometric condition characterizing approximate controllability of $\mathfrak{B}_{\mathcal{D}'_{\mathbb{A}}(\mathbb{R}^{d+1})}(p)$, we expect an algebraic-geometric condition that lies between the two characterizations of controllability and approximate controllability given in this article in Theorems 1.3 and 1.5.

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DEPARTMENT OF MATHEMATICS, LONDON SCHOOL OF ECONOMICS, HOUGHTON STREET, LONDON WC2A 2AE, UNITED KINGDOM.

E-mail address: `sasane@lse.ac.uk`